NULL–CONTROLLABILITY OF ONE–DIMENSIONAL PARABOLIC EQUATIONS

G. ALESSANDRINI AND L. ESCAURIAZA

ABSTRACT. We prove the interior null–controllability of one–dimensional parabolic equations with time independent measurable coefficients.

1. INTRODUCTION

Let us consider the following one–dimensional heat equation with variable coefficients

(1.1)
$$\begin{cases} \partial_x\left(a(x)\partial_x z\right) + b(x)\partial_x z + c(x)z - \rho(x)\partial_t z = f\chi_\omega, & 0 < x < 1 \ , \ 0 < t < T \ , \\ z(0,t) = z(1,t) = 0, & 0 \le t \le T \ , \\ z(x,0) = z_0, & 0 \le x \le 1 \ . \end{cases}$$

Here, $z(x,t)$ is the state and $f(x,t)$ is an interior control that acts on the system over the open set $\omega \subset (0,1)$. The coefficients a, b, c and ρ are assumed to be measurable, bounded and for some $K \ge 1$,

(1.2) $\quad K^{-1} \le \rho(x) \le K, \quad K^{-1} \le a(x) \le K, \ |b(x)| + |c(x)| \le K \ , \ \text{a.e. in } [0,1] \ .$

For any given z_0 in $L^2(0,1)$ and f in $L^2(\omega \times [0,T])$, there is only one solution z to (1.1) in $C([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1))$. The goal of this paper is to analyze the interior null–controllability of (1.1). Specifically, we want to solve the following problem:

Given $T > 0$ and z_0 in $L^2(0,1)$, to find f in $L^2(\omega \times [0,T])$ such that the corresponding solution z to (1.1) satisfies, $z(\cdot, T) \equiv 0$ in $(0,1)$.

In [12], it is shown that the system (1.1) is null–controllable at any positive time, when the coefficients a and ρ are Lipschitz in $[0,1]$. In this reference, the proof of null–controllability is based on an appropriate observability inequality for the adjoint system and it is implied by a global Carleman estimate. When the coefficients are smooth, the observability inequality can be proved introducing Fourier series and using high frequency asymptotic formulae for the eigenvalues of the corresponding Stürm-Liouville problem and classical results on the sums of real exponentials, see [17]. In [9] adopting the approach introduced by D.L. Russel in [19] (the null controllability of the wave operator at large times implies the null controllability of the heat equation at all times) it is shown that the system (1.1) is null controllable, when a and ρ have bounded variation in $[0,1]$.

The main result in this paper is the following.

Theorem 1. *Assume that the coefficients a, b, c and ρ are bounded measurable and satisfy (1.2). Then, (1.1) is null–controllable at time T, for all $T > 0$ and with controls f in $L^2(0, T; H_0^1(\omega))$.*

To prove this result we proceed in the the following way. First, a change of variables shows that the internal controllability of the system (1.1) is equivalent to the same question for a system

$$(1.3) \qquad \begin{cases} \partial_x^2 z - \rho(x)\partial_t z = f\chi_\omega, & 0 < x < 1 \ , \ 0 < t < T \ , \\ z(0, t) = z(1, t) = 0, & 0 \le t \le T \ , \\ z(x, 0) = z_0, & 0 \le x \le 1 \ , \end{cases}$$

where ρ is a new measurable function satisfying (1.2), for some new constant only depending on K, which we shall continue to denote by K.

Then, if we denote by δ the inradius of the open set ω, that is

$$(1.4) \qquad \delta = \sup\{r > 0 | \ \exists \xi \in \omega \ , \ (\xi - r, \xi + r) \subset \omega\} \ ,$$

and if $e_1, e_2, \ldots, e_n \ldots$ and $0 < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_m^2 \ldots$ are respectively the eigenfunctions and eigenvalues of the Stürm-Liouville problem

$$(1.5) \qquad \begin{cases} e'' + \rho(x)\lambda^2 e = 0 \ , \ 0 < x < 1 \ , \\ e(0) = e(1) = 0 \ , \end{cases}$$

we prove the following:

Theorem 2. *Assume that the coefficient ρ satisfies (1.2). Then, there is a constant N, which depends on K and on δ such that the inequality*

$$(1.6) \qquad \sum_{\lambda_k \le \mu} a_k^2 \le Ne^{N\mu} \int_\lambda |\sum_{\lambda_k \le \mu} a_k e_k|^2 \, dx \ ,$$

holds for all $\mu \ge 1$ and all sequences $\{a_k\}$.

The analog of this inequality for the eigenfunctions of the Laplace-Beltrami operator on a compact and *smooth* Riemannian manifold with a possibly nonempty boundary was proved in [14]. There, G. Lebeau and L. Robbiano showed that it implies the interior null-controllability of the heat equation over the manifold by giving an explicit construction of the control function f (See [15, §5] for a more simplified presentation).

The arguments in [14] show that the same iterative method of construction of the control function f given in [14] works for the system (1.3), when Theorem 2 holds. Thus, Theorem 1 follows from Theorem 2.

To prove Theorem 2 we start by following the arguments in [14]. In particular, given $\mu \ge 1$ and a sequence of real numbers $a_1, a_2, \ldots, a_n \ldots$, we set

$$u(x, y) = \sum_{\lambda_k \le \mu} a_k e_k(x) \cosh(\lambda_k y) \ .$$

This function satisfies

$$(1.7) \qquad \begin{cases} \partial_x^2 u + \partial_y(\rho(x)\partial_y u) = 0, & 0 < x < 1 \ , \ y \in \mathbb{R} \ , \\ u(0, y) = u(1, y) = 0, & y \in \mathbb{R} \ , \\ \partial_y u(x, 0) = 0, & 0 < x < 1 \ , \end{cases}$$

and the proof of Theorem 2 is a consequence of a *quantification* of the following *qualitative* result of unique continuation from the boundary:

Assume that u satisfies (1.7) *and* $u(x, 0) \equiv 0$, *when x is in* $\omega \subset (0, 1).$ *Then,* $u \equiv 0$ *in* $[0, 1] \times \mathbb{R}$.

In [14], the one dimensional interval $[0, 1]$ is replaced by a compact and *smooth* manifold M, ∂_x^2 by the corresponding Laplace-Beltrami operator on M and the authors work out the *quantification* of a similar *qualitative* property of boundary unique continuation for the elliptic operator, $\triangle + \partial_y^2$, where $\triangle$ is the Laplace-Beltrami operator on M. To carry out this quantification they use two Carleman inequalities. Those methods require that the elliptic operator involved has Lipschitz second order coefficients and so, they can not be applied to the elliptic operator in (1.7), which has measurable coefficients.

On the other hand, if σ is a 2×2 symmetric and measurable matrix in the plane verifying the ellipticity condition

$$(1.8) \qquad K^{-1}|\xi|^2 \leq \sigma(x, y)\xi \cdot \xi \leq K|\xi|^2, \text{ when } (x, y) \text{ and } \xi \in \mathbb{R}^2 \ ,$$

the weak solutions of the equation

$$(1.9) \qquad \nabla \cdot (\sigma(x, y)\nabla u) = 0 \ ,$$

satisfy the strong unique continuation property:

If a $W_{loc}^{1,2}$-solution of (1.9) *on a connected open set Ω has a zero of infinite order at an interior point, then it must be zero.*

See [2]. This qualitative result of strong unique continuation for uniformly elliptic equations in two independent variables is based on the connection between the solutions of these equations and the theory of quasiregular mappings [5] and on the so-called Ahlfors-Bers representation [1] of such mappings. Here, we describe some quantifications of this qualitative result and apply them to prove the null–controllability property. In particular, a "Hadamard's three circle theorem", Proposition 1, and a "doubling" type property, Proposition 2, adapted to the solutions of (1.9).

In section 2 we recall the results we need from the theory of quasiregular mappings and prove the adapted Hadamard's three circle theorem and doubling property. In section 3 we show how to apply them to prove Theorem 2, also using an estimate of continuation from Cauchy data for solutions of (1.9) Lemma 1, which we adapt from [3]. It may be worth noting that the approach used for the proof of Lemma 1, is based on a variation on the classical principle of majorization by harmonic measure, [20, Chapter VIII, §1, p. 301], which in turn has its roots in arguments due to Carleman [7, p. 3–4].

2. Quantitative estimates of unique continuation with discontinuous coefficients

Throughout the paper, $z = x + iy$, Ω is a simply connected open set in the plane, B_r a circle of radius r centered at the origin, and

$$\partial_{\overline{z}} f = \tfrac{1}{2}\left(\partial_x f + i\partial_y f\right) \ , \ \ \partial_z f = \tfrac{1}{2}\left(\partial_x f - i\partial_y f\right) \ .$$

We shall denote by C constants only depending on K, whereas by N we shall denote constants only depending on K and δ.

When $u \in W_{loc}^{1,2}(\Omega)$ is a weak solution to (1.9), and σ satisfies (1.8) we can associate in a natural fashion, which generalizes the harmonic conjugate, a new function, the so called *stream* function v, which satisfies

$$(2.1) \qquad \nabla v = J\sigma\nabla u$$

almost everywhere in Ω and is a weak solution to

$$(2.2) \qquad \nabla \cdot \left(\frac{\sigma}{\det\sigma}\nabla v\right) = 0, \text{ in } \Omega .$$

Here J denotes the matrix representing a 90° rotation in the plane

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Moreover, letting $f = u + iv$, we have $f \in W_{loc}^{1,2}$ and satisfies

$$(2.3) \qquad \partial_{\overline{z}}f = \mu\partial_z f + \nu\overline{\partial_z f}, \text{ almost everywhere in } \Omega ,$$

where the complex valued functions μ and ν can be explicitly expressed in terms of σ , see [2], and verify

$$(2.4) \qquad |\mu| + |\nu| \le \frac{K-1}{K+1} < 1, \text{ almost everywhere in } \Omega$$

That is, f is a K-quasiregular mapping.

To give an idea of why these results hold, observe that the vector field

$$J\sigma\nabla u$$

is, in the weak sense, curl-free in Ω. To verify that v is a $W_{loc}^{1,2}$-solution of (2.2), observe that, from (2.1), one obtains that the vector field

$$\frac{\sigma}{\det\sigma}\nabla v = J\sigma^{-1}J^t\nabla v$$

is, in the weak sense, divergence-free in Ω.

By the Ahlfors-Bers representation [1] (see also [5] and [4, Chapter II.6, pp. 258–259]), any K-quasiregular mapping f in B_1 can be written as

$$f = F \circ \chi ,$$

where F is holomorphic in B_1 and $\zeta = \chi(z)$ is a K-quasiconformal homeomorphism from B_1 onto B_1, which verifies, $\chi(0) = 0$, $\chi(1) = 1$,

$$(2.5) \qquad C^{-1}|z_1 - z_2|^{\frac{1}{\alpha}} \le |\chi(z_1) - \chi(z_2)| \le C|z_1 - z_2|^{\alpha} , \text{ when } z_1, z_2 \in B_1$$

for some $0 < \alpha < 1$ and $C \ge 1$ depending only on K.

We now recall the Hadamard's three-circle theorem [18].

Theorem 3. *Let F be a holomorphic function of a complex variable in the ball B_{r_2} and $M(r) = \max_{B_r} |F|$. Then, the following is valid for $0 < r_1 \le r \le r_2$,*

$$\log M(r) \le \frac{\log\frac{r_2}{r}}{\log\frac{r_2}{r_1}}\log M(r_1) + \frac{\log\frac{r}{r_1}}{\log\frac{r_2}{r_1}}\log M(r_2) .$$

The meaning of this inequality is that $\log M(r)$ is a convex function of the variable $\log r$.

Let $u \in W_{loc}^{1,2}(B_R)$ be a weak solution to (1.9) and let $f : B_R \longrightarrow \mathbb{C}$ be the associated K-quasiregular mapping. Rescaling (2.5) we have that $f = F \circ \chi$, where F is holomorphic in B_R and $\chi : B_R \longrightarrow B_R$ is a K-quasiconformal homeomorphism, which verifies

$$(2.6) \qquad RC^{-1}|\tfrac{z}{R}|^{\frac{1}{\alpha}} \le |\chi(z)| \le RC|\tfrac{z}{R}|^{\alpha}$$

where C is the same as in (2.5).

Define

$$(2.7) \qquad \mathcal{B}_r = \{z \in B_R : |\chi(z)| < r\}$$

and

$$(2.8) \qquad m(r) = \max_{\mathcal{B}_r} |f(z)| \ , \ \text{when } r < R \ .$$

Then, through the change of coordinates, $\zeta = \chi(z)$, the Hadamard's three circle theorem takes the form: the function $\log m(r)$ is a convex function of $\log r$,

$$(2.9) \quad \log m(r) \le \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}} \log m(r_1) + \frac{\log \frac{r}{r_1}}{\log \frac{r_2}{r_1}} \log m(r_2) \ , \ \text{when } 0 < r_1 \le r \le r_2 < R,$$

and the sets $\mathcal{B}_r$, almost look like balls. In particular,

$$(2.10) \qquad \mathcal{B}_R = B_R \ \ , \ \ B_{R\left(\frac{r}{CR}\right)^{\frac{1}{\alpha}}} \subset \mathcal{B}_r \subset B_{R\left(\frac{Cr}{R}\right)^{\alpha}} \ , \ \ \text{when } r < R$$

and C is the same constant appearing in (2.6). Note incidentally that (2.9) implies a weak unique continuation property, that is, if $m(r_1) = 0$ for some small r_1, then $m(r) = 0$ for all $r < R$.

On the other hand, the difference quotients of convex functions are nondecreasing functions of their arguments. This implies that, if f is not identically zero,

$$\frac{\log m(\frac{r}{2}) - \log m(\frac{r}{4})}{\log \frac{r}{2} - \log \frac{r}{4}} \le \frac{\log m(\frac{R}{2}) - \log m(\frac{R}{4})}{\log \frac{R}{2} - \log \frac{R}{4}} \ , \ \ \text{when } r \le R \ ,$$

and thus,

$$(2.11) \qquad \frac{m(\frac{r}{2})}{m(\frac{r}{4})} \le \frac{m(\frac{R}{2})}{m(\frac{R}{4})} \ , \ \ \text{when } r < R \ .$$

We may prescribe that the the stream function v of u satisfies $v(0) = 0$. We have that $F = u + iv$ is holomorphic in the $\zeta = \xi + i\eta$ coordinates in B_R, hence, solving the Cauchy-Riemann equations,

$$v(\xi, \eta) = \int_0^\eta u_\xi(\xi, s) \, ds - \int_0^\xi u_\eta(t, 0) \, dt \ , \ \text{in } B_R \ .$$

This formula and interior estimates for harmonic functions [11] show that in the ζ-coordinates we have,

$$\|u\|_{L^\infty(B_r)} \le \max_{B_r} |F(\zeta)| \le C\|u\|_{L^\infty(B_{2r})} \ , \ \text{when } r \le \frac{R}{2}$$

where $C > 0$ is an absolute constant. In the z-coordinates, the last inequality reads as

$$(2.12) \qquad \|u\|_{L^\infty(\mathcal{B}_r)} \le \max_{\mathcal{B}_r} |f(z)| \le C\|u\|_{L^\infty(\mathcal{B}_{2r})} \ , \ \text{when } r \le \frac{R}{2} \ ,$$

and from (2.12), (2.9) we obtain:

Proposition 1. *Let $u \in W^{1,2}_{loc}(B_R)$ be a weak solution to (1.9) and let $\mathcal{B}_r$, $0 < r \le R$ be the open sets introduced in (2.7), then we have*
(2.13)
$$\|u\|_{L^\infty(\mathcal{B}_{\frac{r}{2}})} \le C\|u\|^{\theta}_{L^\infty(\mathcal{B}_{r_1})}\|u\|^{1-\theta}_{L^\infty(\mathcal{B}_{r_2})} \ , \ \text{when } r_1 \le r \le r_2 < R \ , \ \theta = \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}} \ .$$

And also, from (2.11):

Proposition 2. *Let u and $\mathcal{B}_r$ be as above. If u is not identically zero, then we have*

$$(2.14) \qquad \frac{\|u\|_{L^\infty(\mathcal{B}_r)}}{\|u\|_{L^\infty(\mathcal{B}_{\frac{r}{2}})}} \leq C \frac{\|u\|_{L^\infty(\mathcal{B}_R)}}{\|u\|_{L^\infty(\mathcal{B}_{\frac{R}{4}})}} \ , \ \ when \ r \leq R \ .$$

These are respectively a Hadamard's three circle theorem and a doubling property adapted to the solution u through the family of "balls" $\mathcal{B}_r$.

3. Proof of Theorem 1

First let us note that, possibly replacing z in (1.1) with $e^{-2Kt}z$, we may assume that c is nonpositive. Introducing $B(x) = \int_0^x \frac{b(s)}{a(s)} ds$, we observe that (1.1) can be rewritten as

$$e^{-B(x)}\partial_x \left(a(x)e^{B(x)}\partial_x z \right) + c(x)z - \rho(x)\partial_t z = f\chi_\omega \ , \ \ \text{in } (0,1) \times (0,T] \ .$$

The solution w to

$$\begin{cases} e^{-B(x)}\frac{d}{dx}\left(a(x)e^{B(x)}\frac{dw}{dx}\right) + c(x)w = 0 \ , \\ w(0) = w(1) = 1 \ , \end{cases}$$

verifies, $0 < w(x) \leq 1$ in $[0,1]$ and replacing z with the new dependent variable $\tilde{z} = z/w$, which we denote again z, we have

$$e^{-B(x)}\partial_x \left(a(x)w^2(x)e^{B(x)}\partial_x z \right) - \rho(x)w^2(x)\partial_t z = w(x)f\chi_\omega \ , \ \ \text{in } (0,1) \times (0,T] \ .$$

Setting

$$L = \int_0^1 a^{-1}(s)w^{-2}(s)e^{-B(s)} ds \ , \ y = \tfrac{1}{L} \int_0^x a^{-1}(s)w^{-2}(s)e^{-B(s)} ds$$

and writing $z(x,t) = \tilde{z}(y,t)$, the new function $\tilde{z}$, which again we rename z, is a solution of a system of the form (1.3).

Considering the associated Stürm-Liouville problem (1.5), we extend the eigenfunctions e_j, $j \geq 1$, to $[-1,1]$ by an odd reflection in 0, similarly we extend ρ by an even reflection in 0. Next, we continue these new functions to all of $\mathbb{R}$ as periodic functions of period 2. The extended ρ verifies (1.2), $e_j \in C^{1,1}(\mathbb{R})$ and $e_j'' + \rho(x)\lambda_j^2 e_j = 0$, almost everywhere in $\mathbb{R}$.

Being the change of variable $y = y(x)$ bi–Lipschitz, with Lipschitz constants which only depend on K, the open set ω is transformed into a new open subset of $(0,1)$ whose inradius is comparable to δ. We continue to denote the transformed set and its inradius by ω and δ, respectively. Also we can assume, up to a translation along the real line, $(-\delta,\delta) \subset \omega \subset (-1,1)$.

Given $\mu \geq 1$ and a sequence of real numbers $a_1, a_2, \ldots a_n, \ldots$, the function

$$u(x,y) = \sum_{\lambda_k \leq \mu} a_k e_k(x) \cosh\left(\lambda_k y\right) \ .$$

verifies

$$(3.1) \qquad \begin{cases} \partial_x^2 u + \partial_y \left(\rho(x)\partial_y u \right) = 0 \ , \ \ \text{in } \mathbb{R}^2 \ , \\ \partial_y u(x,0) = 0 \ , \ \ \text{in } \mathbb{R} \end{cases}$$

and its stream function v can be chosen so that, $v \in W^{1,2}_{loc}(\mathbb{R}^2)$,

$$
\begin{cases} \partial_x v = -\rho(x)\partial_y u , \\ \partial_y v = \partial_x u , \end{cases} \quad \text{and} \quad \begin{cases} \partial_x \left(\frac{1}{\rho(x)}\partial_x v \right) + \partial_y^2 v = 0 , & \text{in } \mathbb{R}^2, \\ v(x,0) = 0, & \text{in } \mathbb{R} . \end{cases}
$$

Let $f = u + iv$, consider the family of "balls" $\mathcal{B}_r$ associated to f in section 2 at scale R and choose $R = 2\,(4C)^{\frac{1}{\alpha}}$, where α and C are the constants in (2.6) and (2.10). With this choice, $\mathcal{B}_{\frac{R}{4}} \supset B_2$. The interior bounds for subsolutions of elliptic equations [11] give

$$
\|u\|_{L^\infty(B_R)} \le \frac{C}{R}\|u\|_{L^2(B_{2R})} \, .
$$

These and the orthogonality of the eigenfunctions e_j, $j \ge 1$ imply that

(3.2)
$$
\frac{\|u\|_{L^\infty(\mathcal{B}_R)}}{\|u\|_{L^\infty(\mathcal{B}_{\frac{R}{4}})}} \le e^{C\mu} \, .
$$

An iteration of (2.14) and (3.2) give

$$
\|u\|_{L^\infty(\mathcal{B}_R)} \le e^{Ck\mu}\|u\|_{L^\infty(\mathcal{B}_{\frac{R}{2^k}})} \, , \quad \text{when } k \ge 1
$$

and from (2.10), there is $k = k(\delta, K)$ such that, $\mathcal{B}_{\frac{R}{2^k}} \subset B_{\frac{\delta}{2}}$. Thus,

(3.3)
$$
\|u\|_{L^\infty(B_1)} \le e^{N\mu}\|u\|_{L^\infty(B_{\frac{\delta}{2}})} \, .
$$

The following inequality, which is an estimate on the continuation from Cauchy data, holds.

Lemma 1. *There are constants $0 < \theta < 1$ and $C > 0$, only depending on K, such that the inequality*

$$
\|u\|_{L^\infty(B_{\frac{r}{2}})} \le Cr^{-\frac{\theta}{2}}\|u(\,\cdot\,,0)\|_{L^2(-r,r)}^\theta \|u\|_{L^\infty(B_{4r})}^{1-\theta}
$$

holds, when $r \le 1$.

The Lemma and (3.3) give

$$
\|u\|_{L^\infty(B_1)} \le Ne^{N\mu}\|u(\,\cdot\,,0)\|_{L^2(-\delta,\delta)} \, ,
$$

which proves Theorem 2.

Proof of Lemma 1. The proof is essentially contained in [3, Theorem 4.5]. For the sake of completeness we summarize here the argument.

Recalling that $f = u + iv$ is analytic in the ζ-variable, we have that $\log|f(\zeta)|$ is subharmonic in B_R and consequently one can verify that $\log|f(z)|$ is a subsolution for an elliptic operator in divergence form E with a matrix of coefficients verifying (1.8). For $r > 0$, let w be the solution to

(3.4)
$$
\begin{cases} Ew = 0 , & \text{in } B_r^+, \\ w = 1 , & \text{in } (-r,r), \\ w = 0 , & \text{in } \partial B_r^+ \setminus (-r,r) . \end{cases}
$$

On ∂B_r^+ we have

$$
\log|f| \le w \log\|u(\,\cdot\,,0)\|_{L^\infty(-r,r)} + (1-w)\log\|f\|_{L^\infty(B_r)}
$$

and the maximum principle implies that the same inequality also holds in B_r^+. The Hölder continuity at the boundary of w and the Harnack's inequality [11] show that there is $\eta \in (0,1)$, which only depends on K, such that $w(z) \ge \eta$ in $B_{\frac{r}{2}}^+$.

Using $v(0) = 0$, (2.1) and interior bounds for elliptic equations [11], we have

$$\|v\|_{L^\infty(B_r)} \le C\|\nabla v\|_{L^2(B_{2r})} \le C\|\nabla u\|_{L^2(B_{2r})} \le C\|u\|_{L^\infty(B_{4r})} .$$

These imply

(3.5) $$\|u\|_{L^\infty(B_{\frac{r}{2}})} \le C\|u(\,\cdot\,,0)\|_{L^\infty(-r,r)}^{\eta}\|u\|_{L^\infty(B_{4r})}^{1-\eta} .$$

For every $\alpha \in (0,1]$, we have the interpolation inequality

(3.6) $$\|\varphi\|_{L^\infty(-r,r)} \le C\left(\|\varphi\|_{L^2(-r.r)}^{\beta}|\varphi|_{C^\alpha(-r,r)}^{1-\beta} + r^{-\frac{1}{2}}\|\varphi\|_{L^2(-r.r)}\right) ,$$

where $\beta = \frac{2\alpha}{1+2\alpha}$, $C > 0$ only depends on α and $|\varphi|_{C^\alpha(-r,r)}$ denotes the standard C^α seminorm. Next, we use the interior Hölder bound for u, [11],

(3.7) $$|u|_{C^\alpha(B_r)} \le Cr^{-\alpha}\|u\|_{L^\infty(B_{4r})} ,$$

with $C > 0$ and $\alpha \in (0,1]$ only depending on K. Combining (3.5) with (3.6) and (3.7), we obtain the thesis with $\theta = \beta\eta$.

The interpolation inequality (3.6) can be proved essentially along the same lines as the interpolation inequalities in [11, §6.8].

□

References

[1] L. Ahlfors, L. Bers, *Riemann's mapping theorem for variable metrics,* Ann. Math. **72** (1960), 265–296.

[2] G. Alessandrini, R. Magnanini, *Elliptic equations in divergence form, geometric critical oints of solutions and Stekloff eigenfunctions,* SIAM J. Math. Anal., **25** n. 5 (1994), 1259–1268.

[3] G. Alessandrini, L. Rondi, *Stable determination of a crack in a planar inhomogeneous conductor,* SIAM J. Math. Anal. **30** n. 2 (1998), 326–340.

[4] L. Bers, F. John, M. Schechter, *Partial Differential Equations,* Interscience, New York, 1964.

[5] L. Bers, L. Nirenberg, *On a representation theorem for linear elliptic systems with discontinuous coefficients and applications* , in Convegno Internazionale sulle Equazioni alle Derivate Parziali, Cremonese, Roma, 1955, 111–138.

[6] C. Castro, E. Zuazua *Concentration and lack of observability of waves in highly heterogeneous media,* Archive Rational Mechanics and Analysis **164** n. 1 (2002), 39–72.

[7] T. Carleman, *Les Fonctions Quasi Analytiques,* Gauthier–Villars, Paris 1926.

[8] E. Fernández-Cara, E. Zuazua, *The cost of approximate controllability for heat equations: The linear case,* Advances Diff. Eqs. **5** (4-6) (2000), 465–514.

[9] ———, *On the null controllability of the one-dimensional heat equation with BV coefficients,* Computational and Applied Mathematics **21** (1) (2002), 167–190.

[10] A. V. Fursikov, O. Yu. Imanuvilov, *Controllability of evolution equations,* Lecture Notes Series **34**, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University (1996).

[11] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* 2nd ed., Springer-Verlag, Berlin-Heildeberg-New York-Tokyo, 1983.

[12] O. Yu Imanuvilov, M. Yamamoto, *Carleman estimate for a parabolic equation in Sobolev spaces of negative order and its applications,* in Control of Nonlinear Distributed Parameter Systems, G. Chen et al. eds., Marcel-Dekker (2000), 113–137.

[13] E.M. Landis, O.A. Oleinik, *Generalized analyticity and some related properties of solutions of elliptic and parabolic equations,* Russian Math. Surv. **29** (1974), 195–212.

[14] G. Lebeau, L. Robbiano, *Contrôle exact de l'équation de la chaleur,* Commun. Partial Differ. Equ. **20** (1995), 335-356.

[15] G. Lebeau, E. Zuazua, *Null controllability of a system of linear thermoelasticity,* Archive for Rational Mechanics and Analysis **141** (4) (1998), 297–329.

[16] F.H. Lin, *A uniqueness theorem for parabolic equations,* Comm. Pure Appl. Math. **42** (1988), 125–136.

[17] A. López, E. Zuazua, *Uniform null-controllability for the one-dimensional heat equation with rapidly oscillating periodic density,* Ann. I.H.P. - Analyse non linéaire **19**, 5 (2002), 543–580.

[18] A.I. Markushevich, *Theory of Functions of a Complex Variable,* Prentice Hall, Englewood Cliffs, NJ, 1965.

[19] D.L. Russel, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations,* Studies in Appl. Math. **52** (1973), 189–221.

[20] M. Tsuji, *Potential Theory in Modern Function Theory,* Maruzen, Tokyo 1959.

[21] I.N. Vekua, *Generalized Analytic Functions,* Pergamon, Oxford 1962.

(G. Alessandrini) Dipartimento di Matematica e Informatica, Università degli Studi di Trieste, Via Valerio, 12/b, 34127 Trieste, Italy.

E-mail address: `alessang@univ.trieste.it`

(L. Escauriaza) Universidad del País Vasco / Euskal Herriko Unibertsitatea, Dpto. de Matemáticas, Apto. 644, 48080 Bilbao, Spain.

E-mail address: `luis.escauriaza@ehu.es`